International Journal of Theoretical Physics, Vol. 43, No. 2, February 2004 (© 2004)

Born–Infeld Action on Discrete Spaces

Liangzhong Hu,[1,3] Liangyou Hu,[2] and Adonai S. Sant'Anna[1]

We apply Connes' noncommutative geometry to a finite *n*-point space. The explicit Born-Infeld actions on this *n*-point space and *n* copies of a manifold are obtained.

KEY WORDS: Born–Infeld action; Fredholm module; discrete space.

1. INTRODUCTION

In recent years, the continuum Born–Infeld theory (Born and Infeld, 1934) in its commutative and noncommutative settings has become relevant in the description of *D*-brain dynamics (see, for example, Seiberg and Witten, 1999; Tseytlin, 2000). The Born–Infeld actions on finite group spaces was constructed in Aschieri *et al*. (2003).

In this paper, we apply Connes' noncommutative geometry (Connes, 1985, 1994) to a finite *n*-point space. By explicit Born–Infeld actions on this *n*-point space, *n* copies of a manifold are obtained.

2. DIFFERENTIAL CALCULUS ON *n*-POINT SPACE

We briefly review the differential calculus on a *n*-point space. More detailed account of the construction can be found in Cammarata and Coquereaux (1995), Dimakis and Müller-Hoissen (1994a,b), and Hu and Sant'Anna (2002, 2003).

Let M be a space of n points $i_1, \ldots, i_n (n < \infty)$, and $\mathcal{A}$ the algebra of complex functions on M with $(fg)(i) = f(i)g(i)$. Let $p_i \in \mathcal{A}$ defined by

$$p_i(j) = \delta_{ij}. \quad (1)$$

[1] Department of Mathematics, Federal University of Paraná, C.P. 019081, Curitiba, PR, 81531-990, Brazil.
[2] Dongbei University of Finance and Economics, Dalian 116025, People's Republic of China.
[3] To whom correspondence should be addressed at Department of Mathematics, Federal University of Paraná, C.P. 019081, Curitiba, PR, 81531-990, Brazil; e-mail address: hu@mat.ufpr.br.

0020-7748/04/0200-0315/0 © 2004 Plenum Publishing Corporation

It follows that p_i is a projector in $\mathcal{A}(i = 1, \ldots, n)$. Each $f \in \mathcal{A}$ can be written as

$$f = \sum_i f(i) p_i,$$

where $f(i) \in \mathbf{C}$, a complex number. The algebra $\mathcal{A}$ can be extended to a universal differential algebra $\Omega(\mathcal{A}) = \oplus_{r=0}^{\infty} \Omega_r(\mathcal{A})$ (where $\Omega^0(\mathcal{A}) = \mathcal{A}$) via the action of a linear operator $d : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A})$ satifying

$$d1 = 0,\, d^2 = 0,\, d(\omega_r \omega') = (d\omega_r)\omega' + (-1)^r \omega_r d\omega',$$

where $\omega_r \in \Omega^r(\mathcal{A})$. 1 is the unit in $\Omega(\mathcal{A})$.

Let $\varepsilon = \mathcal{A}^m$ be a free $\mathcal{A}$-module. A connection on ε is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that

$$\nabla(\psi_a) = (\nabla\psi)_a + \psi \otimes da, \tag{2}$$

for all $\psi \in \mathcal{E}$, $a \in \mathcal{A}$.

Any connection on $\mathcal{E}$ is of the form $\nabla = d + A$ with $A^* = -A$. A is called a connection 1-form. We can regard A as an element of $M_m(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$. Here $M_m(\mathcal{A})$ is a $m \times m$ matrix algebra over $\mathcal{A}$. A can be written as $A = \sum_{i,j} A_{ij} p_i \, dp_j$ with $A_{ij} \in M_m$ $(\mathbf{C})$, a $m \times m$ complex matrix, and $A_{ii} = 0$, a $m \times m$ zero matrix. From $A^* = -A$, we have

$$A_{ij}^* = A_{ji}. \tag{3}$$

Let $\mathtt{G} \subset \mathrm{End}_{\mathcal{A}}(\varepsilon) = M_m(\mathcal{A})$ be a gauge group of ε. Then $\mathtt{G} = \sum_i \mathtt{G}_i p_i$ with $\mathtt{G}_i \subset M_m(\mathbf{C})$. Notice that

$$\mathtt{G}_1 = \mathtt{G}_2 = \cdots = \mathtt{G}_n = G. \tag{4}$$

The connection 1-form A satisfies

$$A' = g\, A g^{-1} + g\, dg^{-1}. \tag{5}$$

Here $g = \sum_i g_i p_i \in \mathtt{G}$, and $g_i \in \mathtt{G}_i = G$.

The curvature of ∇ reads

$$\Theta = dA + A^2. \tag{6}$$

Θ transforms in the usual way, $\Theta' = g\Theta g^{-1}$. One has $\Theta^* = \Theta$.

Θ satisfies the Bianchi identity:

$$d\Theta + A\Theta - \Theta A = 0.$$

3. FROM FREDHOLM MODULE TO BORN–INFELD ACTION ON *M*

One of the basic ideas in Connes' noncommutative geometry is the Fredholm module (Connes, 1994, and references therein). Applying the Fredholm module to the universal algebra $\Omega(\mathcal{A})$ discussed in the previous section, one can obtain a more useful graded differential algebra on the finite space M.

The Fredholm module $(\mathcal{A}, \mathcal{H}, D)$ is composed as the following (Hu, 2000; Hu and Sant'Anna, 2002, 2003): $\mathcal{A}$ is the algebra on M defined in the previous section. $\mathcal{H}$ is a n-dimensional linear space over the complex field $\mathbf{C}$. The action of $\mathcal{A}$ on $\mathcal{H}$ is given by

$$\pi(f) = \begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f(n) \end{pmatrix}$$

with $f \in \mathcal{A}$. D is a Hermitian $n \times n$ matrix with $D_{ij} = \overline{D}_{ji}$. The following equality defines an involutive representation of $\Omega(\mathcal{A})$ in $\mathcal{H}$,

$$\pi(da) = [D, \pi(a)], \tag{7}$$

where $a \in \mathcal{A}$. To ensure the differential d satisfies

$$d^2 = 0, \tag{8}$$

one has to impose the following condition on D,

$$D^2 = \mu^2 \mathbf{I}, \tag{9}$$

where μ is a real constant and $\mathbf{I}$ is the $n \times n$ identity matrix. Since the diagonal elements of D commute exactly with the action of $\mathcal{A}$, we can ignore the diagonal elements of D, i.e.,

$$D_{ii} = 0. \tag{10}$$

The projector p_i can be expressed as a $n \times n$ matrix,

$$(\pi(p_i))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i}. \tag{11}$$

From Eq. (7) and (11), it follows that

$$(\pi(p_i\, dp_j))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta j} D_{ij}. \tag{12}$$

The connection matrix H on M is given by

$$H_{ij} = D_{ij}(A_{ij} + \mathbf{1}). \tag{13}$$

Here $\mathbf{1}$ is the identity in the gauge group G, where G is defind in Eq. (4). One can find that H_{ij} is a $m \times m$ complex matrix with $H_{ij}^* = H_{ji}$. This means that $H = (H_{ij})$ is a $n \times n$ Hermitian matrix with its elements $m \times m$ submatrices. The diagonal elements of H satisfy

$$H_{ii} = 0. \tag{14}$$

From (5) and (13), the transformation rule of H_{ij} reads

$$H'_{ij} = g_i H_{ij} g_j^{-1}. \tag{15}$$

From (6), the curvature matrix $\pi(\Theta)$ reads

$$\pi(\Theta) = H^2 - \mu^2 I, \tag{16}$$

where $I = (I_{ij}) = (\delta_{ij}\mathbf{1})$. The transformation rule of $\pi(\Theta_{ij})$ satisfies

$$\pi(\Theta'_{ij}) = g_i \pi(\Theta_{ij}) g_j^{-1}. \tag{17}$$

We recall the continuum p-dimensional Born–Infeld action for nonlinear electrodynamics (Born and Infeld, 1934) in flat space is

$$S = \int_{V^p} d^p x \sqrt{\det(\delta_{\mu\nu} + F_{\mu\nu})}, \tag{18}$$

where F is the field strength. The action (18) can be generalized to the non-Abelian case. Then the determinant in (18) is not a number. We can define its absolute value $|\det|$ as the positive square root in $\sqrt{\det \det^\dagger}$. The generalized Born–Infeld action is

$$S = \int_{V^p} d^p x Tr \sqrt{|\det(\delta_{\mu\nu} + F_{\mu\nu})|}. \tag{19}$$

The trace can be symmetrized (Tseytlin, 2000, and references therein).

Now we construct the non-Abelian Born–Infeld action on the finite n-point space M. The analogue of $\delta_{\mu\nu} + F_{\mu\nu}$ becomes

$$T_{ij} = \mathbf{1}\delta_{ij} + \pi(\Theta_{ij}) = (1 - \mu^2)\mathbf{1}\delta_{ij} + (H^2)_{ij}, \tag{20}$$

and transforms under the gauge transformation in the same way as $\pi(\Theta_{ij})$:

$$T'_{ij} = g_i T_{ij} g_j. \tag{21}$$

The determinant of T is unchanged under the gauge transformation:

$$\det T' = \det T. \tag{22}$$

Therefore, the non-Abelian Born–Infeld action on M reads

$$\begin{aligned} S &= Tr\sqrt{|\det(T_{ij})|} \\ &= Tr\sqrt{|\det((1 - \mu^2)\mathbf{1}\delta_{ij} + (H^2)_{ij})|} \end{aligned} \tag{23}$$

Example 1. When $\mu^2 = 1$, $S = Tr|\det(H_{ij})|$.

Example 2. The $U(1)$ Born–Infeld action on M reads

$$S = \sqrt{|\det(T_{ij})|}$$
$$= \sqrt{|\det((1-\mu^2)\delta_{ij} + (H^2)_{ij})|} \tag{24}$$

When $\mu^2 = 1$, $S = |\det(H_{ij})|$. Here H_{ij} is a complex number.

4. BORN–INFELD ACTION ON *n* COPIES OF A MANIFOLD

Let V be an oriented and connected smooth manifold and M, as the previous sections, a n-point space. We see that $V \times M$ is a disconnected manifold consisting of n copies of V.

We briefly review the gauge theory on n copies of V (Hu and Sant'Anna, 2002, 2003). Denote the differential on M by d_f, i.e., the differential d in previous sections is replaced by d_f. Let d_s be the usual differential on V, and d the total differential on $V \times M$. It follows that

$$d = d_s + d_f. \tag{25}$$

The nilpotency of d requires that

$$d_s d_f = -d_f d_s. \tag{26}$$

The connection A has a usual differential degree and a finite-difference degree (α, β) adding up to 1:

$$A^{(1,0)} = \sum_i A_i p_i. \tag{27}$$

It is the continuous part of A. A_i is a Lie algebra valued 1-form on V_i and $A_i^* = -A_i$.

$$A^{(0,1)} = \sum_{i,j} A_{ij} p_i d_f p_j. \tag{28}$$

It is the connection 1-form on M, and is well studied in the previous section.

We see that the curvature $\pi(\Theta)$ has a usual differential degree and a finite-difference degree (α, β) adding up to 2:

$$\Theta_{ij}^{(2,0)} = F_i \delta_{ij}, \tag{29}$$

Here F_i is the curvature of A_i, $F_i = d_s A_i + A_i \wedge A_i$.

$\Theta_{ii}^{(2,0)}$ obeys the transformation rule,

$$\Theta_{ii}^{\prime(2,0)} = g_i \Theta_{ii}^{(2,0)} g_i^{-1},$$

where $g_i \in \mathtt{G}_i$, and $\mathtt{G}_i$ is the gauge group on V_i. Let G be the gauge group on V, then $G = \mathtt{G}_1 = \ldots = \mathtt{G}_n$. $\Theta^{(2,0)}$ is the continuous part of the field strength.

Next, we look at the component $\Theta^{(1,1)}$ of bi-degree (1,1):

$$\Theta_{ij}^{(1,1)} = d_s H_{ij} + A_i H_{ij} - H_{ij} A_j. \tag{30}$$

One can find that $\Theta_{ij}^{(1,1)}$ transforms as the following:

$$\Theta_{ij}^{\prime(1,1)} = g_i \Theta_{ij}^{(1,1)} g_j^{-1}.$$

$\Theta^{(1,1)}$ corresponds to the interaction between V and M.

We can define a covariant derivative of H_{ij} as

$$D_\mu H_{ij} = \partial_\mu H_{ij} + A_{i\mu} H_{ij} - H_{ij} A_{j\mu}. \tag{31}$$

Therefore $\Theta_{ij}^{(1,1)} = D_\mu H_{ij}\, dx^\mu$. Here the Einstein sum convention for the indice μ is adopted.

Finally, we have the component $\Theta^{(0,2)}$ of degree (0,2):

$$\Theta^{(0,2)} = H^2 - \mu^2 \mathbf{I}, \tag{32}$$

with

$$\Theta_{ij}^{\prime(0,2)} = g_i \Theta_{ij}^{(0,2)} g_j^{-1}.$$

$\Theta_{ij}^{(0,2)}$ corresponds to the field strength on the finite space M.

Now we construct the Born–Infeld action on the n copies of V. The curvature $\pi(\Theta)$ can be formally written as

$$\pi(\Theta) = \begin{pmatrix} \Theta^{(2,0)} & \Theta^{(1,1)} \\ \Theta^{(1,1)} & \Theta^{(0,2)} \end{pmatrix} \tag{33}$$

Using a trick in Aschieri *et al.*, (2003), we consider the algebraic identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}$$

This implies

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = [\det(A - BD^{-1}C)][\det D] \tag{34}$$

Let

$$\begin{aligned} K_{\mu\nu} &= [A - B(D^{-1})C]_{\mu\nu} \\ &= g_{\mu\nu} \begin{pmatrix} \mathbf{1} & 0 & \cdots & 0 \\ 0 & \mathbf{1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathbf{1} \end{pmatrix} \end{aligned}$$

$$+\begin{pmatrix} F_{1\mu\nu} & 0 & \cdots & 0 \\ 0 & F_{2\mu\nu} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_{n\mu\nu} \end{pmatrix} - D_\mu H T^{-1} D_\nu H, \tag{35}$$

where $g_{\mu\nu}$ is the metric on the manifold V, and $T = (T_{ij})$ is defined in (20). Notice that all the matrices in (35) have n rows and n columns with their elements $m \times m$ submatrices. One can find that $\det(K_{\mu\nu})$ is unchanged under the gauge transformation:

$$\det(K'_{\mu\nu}) = \det(K_{\mu\nu}). \tag{36}$$

Hence, the Born–Infeld action on $V \times M$ reads

$$S = \int_{V^p} d^p x Tr \sqrt{|\det(K_{\mu\nu}) \det(T_{ij})|}. \tag{37}$$

REFERENCES

Aschieri, P., Castellani, L., and Isaev, A. P. (2003). Discretized Yang–Mills and Born–Infeld actions on finite group geometries. *International Journal of Modern Physics* A **18**, 3555.

Born, M. and Infeld, L. (1934). Foundations of the new field theory. *Proceedings of Royal Society* A **144**, 425.

Cammarata, G. and Coquereaux, R. (1995). Comments about Higgs fields, noncommutative geometry and the standard model. In *Lecture Notes in Physics, Vol. 469*, Springer, Berlin, pp. 27–50. (hep-th/9505192).

Connes, A. (1994). Noncommutative differential geometry. *Publ. Math. I. H. E. S.* **62**, 257.

Connes, A. (1994). *Noncommutative Geometry*, Academic Press, New York.

Dimakis, A. and Muller-Hoissen, F. (1994a). Differential calculus and gauge theory on finite sets. *Journal of Physics A: Mathematical and General* **27**, 3159.

Dimakis, A. and Muller-Hoissen, F. (1994b). Discrete differential calculus: Graphs, topologies and gauge theory. *Journal of Mathematical Physics* **35**, 6703.

Hu, L.-Z. (2000). *$U(1)$ Gauge Theory Over Discrete Space-Time and Phase Transitions*. (hep-th/0001148).

Hu, L.-Z. and Sant'Anna, S. S. (2002). Connes' spectral triple and U(1) gauge theory on finite sets. *Journal of Geometry and Physics* **42**, 296.

Hu, L.-Z. and Sant'Anna, S. S. (2003). Gauge theory on a discrete noncommutative space. *International Journal of Theoretical Physics* **42**, 635.

Seiberg, N. and Witten, E. (1999). String theory and noncommutative geometry. *The Journal of High Energy Physics* **9909**, 032.

Tseytlin, A. A. (2000). Born–Infeld action, supersymmetry and string theory. In *The Many Faces of the Superworld, Yuri Golfand Memorial Volume*, M. A. Shifman, ed., World Scientific, Singapore, 417 pp.